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DEFINING EQUATIONS FOR CYCLIC PRIME COVERS OF THE RIEMANN SPHERE

BY

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ABSTRACT

We determine a method to find explicit defining equations for each compact Riemann surface which admits a cyclic group of automorphisms C_p of prime order p such that the quotient space has genus 0.

1. Introduction

An open problem in the theory of compact Riemann surfaces is to find a defining equation for such a surface X given a uniformizing subgroup $\Lambda \leq \mathrm{PSL}(2,\mathbb{R})$ for X. In general, when the genus g satisfies $g \geq 2$, it is not at all clear how one could approach this problem. However, under the assumption that the surface admits automorphisms, we can use Galois theory to help us find defining equations; see, for example, [10] and [4]. We shall consider this problem for each compact Riemann surface X which admits an automorphism group C_p of prime order p such that the quotient space X/C_p has genus 0. This is a generalization of the case where X is a hyperelliptic Riemann surface — a surface which admits a cyclic 2 cover of the Riemann sphere. We briefly describe how the equations for a hyperelliptic surface will be found.

If X is a hyperelliptic surface, then X admits a cyclic 2 group of automorphisms C_2 such that X/C_2 has genus 0 which is normal in G, the full automorphism group of X. Since the quotient group G/C_2 acts on the quotient space X/C_2, if we identify X/C_2 with the Riemann sphere, then G/C_2 is identified with a finite group of automorphisms of the Riemann sphere. This group will

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act on the branch points of the quotient map $\pi_{C_2}\colon X \to X/C_2$ (in general, if S is a space on which a group G acts, we let $\pi_G\colon S \to S/G$ denote the quotient map). Kummer theory implies there exists an affine model of the form $y^2 = q(x)$ for some polynomial $q(x)$ and in fact q can be chosen so that the branch points of the quotient map $\pi_{C_2}\colon X \to X/C_2$ are the zeros of the polynomial q. These facts impose enough restrictions on the equation to obtain a defining model for X depending on G.

The obvious generalization is to find defining equations for compact Riemann surfaces which admit a cyclic prime p cover of the Riemann sphere. Two immediate problems arise for $p \geq 3$. Similar to the hyperelliptic case, there is a model of the form $y^p = q(x)$. However, for a hyperelliptic surface, the multiplicities of the linear factors of the polynomial $q(x)$ can only be 1. For $p \geq 3$, the multiplicities can be any n for $1 \leq n \leq p-1$. This means the method we develop will have to determine not only the zeros of the polynomial, but also the multiplicities of the linear factors. The second problem is that for $p \geq 3$, the cyclic prime subgroup C_p is not necessarily normal in the full group of automorphisms of X.

If we assume C_p is normal in G, the full automorphism group of X, in order to put further restrictions on $q(x)$, we can consider the action of G/C_p on the quotient space X/C_p as we did with the hyperelliptic case. More generally, however, we will start with a model of the form $y^p = q(x)$ and, if $N \leq G$ with C_p normal in N, consider the action of N/C_p on the quotient space X/C_p. As we shall see in Section 8, the method produced can be used to derive some very classical models. Before we move on, the following definitions will be very useful.

Definition 1.1: Suppose X is a compact Riemann surface admitting a cyclic prime group of automorphisms C_p of order p with the property that X/C_p has genus 0. Then we call X a cyclic p-gonal surface and C_p a p-gonal group for X.

The outline of our work is as follows. We start in Section 2 by gathering the necessary results from uniformization. Following this, as our reasoning suggests, one key part of the problem is to classify each group N with the following properties for each prime p.

(i) N has a normal cyclic subgroup C_p of prime order p.
(ii) N/C_p is isomorphic to a finite group of automorphisms of the Riemann sphere.

We shall temporarily refer to such a group as a **p-gonal normal overgroup** of C_p, though we shall refine this definition in Section 3. For the hyperelliptic case, a classification of these groups can be found in [1]. For the general case (in fact for general n), all such groups were classified in [7]. Since it will be relevant to our construction and for completeness, we summarize the techniques which can be used to determine these groups in Section 3 giving full presentations for all such groups in Appendix B. Next, in Section 4, we shall describe the structure of the corresponding quotient maps and the discrete groups associated to them.

Once all groups have been described, we shall approach the problem of finding defining equations for cyclic p-gonal surfaces (for some related results using function fields and Teichmüller theory, see [7] and [5] respectively). Since our goal is to use uniformization to provide explicit equations for all cyclic p-gonal surfaces, we outline in Section 5 the known results which explain how uniformization and Galois theory can be used to find a general form for a defining equation for X. Following this, in Section 6, we shall determine the zeros of the polynomial $q(x)$. This knowledge will be sufficient to present complete results for the case where X is hyperelliptic. Next, in Section 7, we shall determine the multiplicities of the linear factors of $q(x)$. The main result we obtain is Theorem 7.5 and is an extension to those found in [11] and [4]. Finally, to illustrate the use of these results, we present explicit examples in Section 8.

2. Uniformization and Fuchsian groups

A compact Riemann surface X of genus $g \geq 2$ can be realized as a quotient of the upper half plane $\mathbb{H}/\Lambda$, where Λ is a torsion free Fuchsian group called a **surface group** for X. Under such a realization, a group G acts as a group of automorphisms on X if and only if $G = \Gamma/\Lambda$ for some Fuchsian group Γ containing Λ as a normal subgroup of index $|G|$. We call Γ the Fuchsian group corresponding to G and, if Λ has been fixed, G the automorphism group corresponding to Γ. If G is group of automorphisms of X with surface group Λ and Γ is the Fuchsian group corresponding to G, we identify the orbit spaces $\mathbb{H}/\Gamma$ and X/G and the quotient map $\pi_G\colon X \to X/G$ is branched over the same points as $\pi_\Gamma\colon \mathbb{H} \to \mathbb{H}/\Gamma$ with the same ramification indices as illustrated in Figure 1.

We define the signature of a Fuchsian group Γ to be the tuple $(g; m_1, m_2, \ldots, m_r)$, where the quotient space $\mathbb{H}/\Gamma$ has genus g and the quotient map π_Γ branches over r points with ramification indices m_i for $1 \leq i \leq r$. The signature of Γ also provides information regarding a presentation for Γ and the local behavior of the elements with fixed points (see [2], Theorem 3.21 for a detailed exposition of condition (v)).

$$\mathbb{H} \xrightarrow{\pi_\Lambda} \mathbb{H}/\Lambda = X \xrightarrow{\pi_G} \mathbb{H}/\Gamma = X/G, \qquad \pi_\Gamma : \mathbb{H} \to \mathbb{H}/\Gamma$$

Figure 1. Holomorphic quotient maps and surface identifications.

Theorem 2.1: *If Γ is a Fuchsian group with signature $(g; m_1, \ldots, m_r)$, then there exist group elements $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots c_r \in \mathrm{PSL}(2, \mathbb{R})$ such that:*

(i) $\Gamma = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots c_r \rangle$.

(ii) *Defining relations for Γ are*

$$c_1^{m_1}, c_2^{m_2}, \ldots, c_r^{m_r}, \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} c_j.$$

(iii) *Each elliptic element (the elements of finite order) lies in a unique conjugate of $\langle c_i \rangle$ for suitable i. Furthermore, the cyclic groups $\langle c_i \rangle$ are self-normalizing in Γ.*

(iv) *Each elliptic element of Γ has a unique fixed point in $\mathbb{H}$. All other elements (the hyperbolic elements) act fixed point freely on $\mathbb{H}$.*

(v) *For $1 \leq i \leq r$, the elliptic generators $c_1, \ldots, c_r$ of Γ of orders $m_1, \ldots, m_r$ respectively can be chosen so that c_i is described by $z \to e^{-2\pi i/m_i} z$ near its fixed point. We call $e^{2\pi i/m_i}$ the rotational constant of c_i.*

We call a set of elements of Γ satisfying Theorem 2.1 **canonical generators** for Γ. Notice that if Γ is a surface group for a surface of genus g, since it is torsion free, it must have signature $(g; -)$.

Given a Fuchsian group Γ with signature $(g; m_1, \ldots m_r)$, to find the normal subgroups of finite index, we consider the possible epimorphisms from Γ onto different finite groups. Given such an epimorphism $\varrho\colon \Gamma \to G$, by knowing explicitly the images of the elliptic generators of Γ, we can determine the signature of the normal subgroup $\mathrm{Ker}(\varrho)$. In the special case where $\mathrm{Ker}(\varrho)$ is torsion free, we call ϱ a **surface kernel epimorphism**. The general method is outlined in the following result.

Proposition 2.2: *Let Γ be a Fuchsian group with signature $(g_\Gamma; m_1, \ldots, m_r)$ and $\Lambda \leq \Gamma$ a normal subgroup of finite index N such that $c_i\Lambda$ has order t_i in the quotient group Γ/Λ. Then the orbit genus g_Λ of Λ is given by*

$$g_\Lambda - 1 = N(g_\Gamma - 1) + \frac{N}{2} \sum_{i=1}^{r} \Big(1 - \frac{1}{t_i}\Big),$$

and the periods of Λ *are* $f_{i,j} = m_i/t_i$, $1 \leq j \leq N/t_i$, $1 \leq i \leq r$, *where* $f_{i,j} = 1$ *are deleted.*

Proof: This is a special case of Theorem 1, [9]. ∎

3. Necessary finite group theory

For this section, we assume that p is an odd prime. For similar results on the hyperelliptic case, we refer the interested reader to [1] and for a detailed examination of the case for general n, to [7].

By definition, a p-gonal normal overgroup G of a p-gonal surface X will contain a cyclic p subgroup C_p which is normal and such that the quotient space X/C_p has genus 0. Since the group $K = G/C_p$ acts on the quotient space X/C_p, it follows that K is a finite group of automorphisms of the Riemann sphere. All such groups are well known and we tabulate them in Table 1. The branching data is a vector whose length is the number of branch points of the map $\pi_K\colon \Sigma \to \Sigma/K$ and whose entries are the orders of the branch points. It follows that any p-gonal normal overgroup must satisfy the short exact sequence:

$$1 \to C_p \to G \to K \to 1$$

Figure 2. Short exact sequence for normal p-gonal overgroups.

Here, K is a finite group of automorphisms of the Riemann sphere.

Group	*BranchingData*
C_n	(n, n)
D_n	$(2, 2, n)$
A_4	$(2, 3, 3)$
S_4	$(2, 3, 4)$
A_5	$(2, 3, 5)$

Table 1. Groups of automorphisms of the Riemann sphere and branching data.

To find the solutions to this short exact sequence, we can consider the map $\mathcal{A}\colon K \to \mathrm{Aut}(C_p) = C_{p-1}$ determined by the action of K on C_p. We shall call this the **action map** of G. If K is one of the three groups of automorphisms of the platonic solids, there are few choices for the kernel of $\mathcal{A}$. Moreover, if $p > 5$ for A_5 or $p > 3$ for A_4 and S_4, since $(p, |K|) = 1$, it is a consequence of the Schur–Zassenhaus Theorem that G is a semi-direct product of K and C_p. This determines all possibilities for G under these conditions. The remaining solutions, when $p = 2$, 3 or 5, can be found by looking through the library of small groups in the GAP database, [7].

When K is a cyclic group, then either G is a split extension of C_p by K or G is cyclic. When K is dihedral of order $2n$, there are up to four solutions if $|K|$ is divisible by 4 and up to three solutions else (depending upon whether p and n are coprime). We tabulate all groups with full presentations in Table 5 in Appendix B. We also include the kernel of the action map $\mathcal{A}$.

Our summary suggests that given a cyclic p-gonal surface X, there are a number of different possible p-gonal normal overgroups which can act on X. We shall explore the different possibilities for these groups and state conditions under which we can make a well-defined unique choice for G (so we can henceforth refer to G as *the* p-gonal normal overgroup of X).

First, for a fixed p-gonal group C_p, if $N_{\mathrm{Aut}(X)}(C_p)$ denotes the normalizer of C_p in $\mathrm{Aut}(X)$, then any group H with $C_p \leq H \leq N_{\mathrm{Aut}(X)}(C_p)$ will also be a p-gonal normal overgroup for X. To avoid this redundancy, given a surface X and a fixed p-gonal group C_p, when building equations for X, we shall just consider the group $N_{\mathrm{Aut}(X)}(C_p)$. The second problem which could occur is that different p-gonal subgroups could have different normalizers, and consequently could be contained in different p-gonal normal overgroups up to isomorphism. However, the following result guarantees that this will never happen.

Proposition 3.1: *If* $\mathrm{Aut}(X)$ *is the full automorphism group of* X*, then there exists a unique conjugacy class of* p*-gonal groups.*

Proof: This is the main result of [5]. ∎

With these results in consideration, we redefine the term p-gonal normal overgroup and fix some further notation and terminology.

Definition 3.2: If X is a p-gonal, we define *the* p-gonal normal overgroup of X to be the group $G = N_{\mathrm{Aut}(X)}(C_p)$ considered as an abstract group where C_p is some p-gonal group of X. If C_p is a p-gonal group of X and $G = N_{\mathrm{Aut}(X)}(C_p)$ considered as a subgroup of $\mathrm{Aut}(X)$, we define G to be *the* p-gonal normal overgroup of C_p. We define the group $K = G/C_p$ to be the **sphere group** of X.

Henceforth, we shall assume that given a p-gonal surface X, C_p denotes some fixed p-gonal group for X and G is its p-gonal normal overgroup in $\mathrm{Aut}(X)$. We shall let Λ denote some fixed surface group for X and Γ_p and Γ denote Fuchsian groups with $\Gamma/\Lambda = G$ and $\Gamma_p/\Lambda = C_p$. Finally, we let $\varrho\colon \Gamma \to G$ and $\chi\colon \Gamma \to K$ denote fixed epimorphisms with kernels Λ and Γ_p respectively. We shall explore the implications of these choices and how they effect the equations in Section 5.

4. Group signatures and ramification points

For a p-gonal surface X, since the map $\pi_{C_p}\colon X \to X/C_p$ is a Galois cover of the Riemann sphere of degree p, at any point it will either be totally ramified of order p, or unramified. It follows that Γ_p with $\Gamma_p/\Lambda = C_p$ has signature $(0; \underbrace{p, \ldots, p}_{R \text{ times}})$ for some integer $R > 2$. In the previous section we found all possible p-gonal normal overgroups. Given such a G, we shall now determine the signature for Γ. First note that after appropriate group and surface identifications, we get the tower of covers illustrated in Figure 3. To find the possible signatures for Γ, we use the fact that we know complete branching data of the maps π_K and π_{C_p}. It is then a simple matter of determining whether or not any branch points of π_{C_p} coincide with any ramification points of π_K. We summarize below.

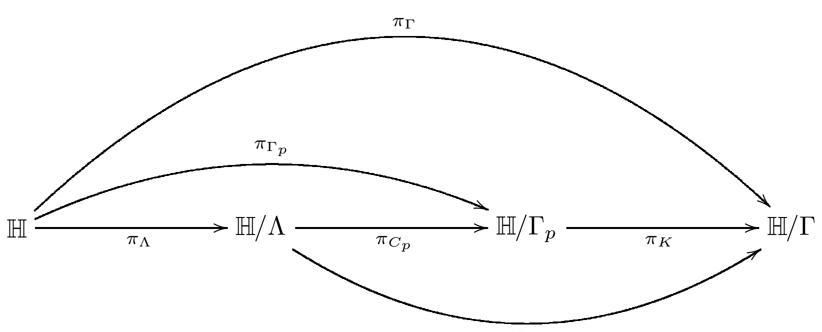

Figure 3. Holomorphic quotient maps and surface identifications.

Proposition 4.1: *The possible signatures for Γ depend upon the sphere group K and are calculated as follows.*

(i) *If $K \neq C_n$ and (m_1, m_2, m_3) is the branching data of the quotient map π_K, the signature of Γ is $(0; am_1, bm_2, cm_3, \underbrace{p, \ldots, p}_{s\text{times}})$ where a, b and c are either 1 or p depending upon whether any branch points of π_{C_p} coincide with ramification points of π_K. For such a Γ, the signature of Γ_p is $(0; \underbrace{p \ldots, p}_{r \text{ times}})$*

where

$$r = s|K| + \frac{(a-1)|K|}{(p-1)m_1} + \frac{(b-1)|K|}{(p-1)m_2} + \frac{(c-1)|K|}{(p-1)m_3}.$$

(ii) *If $K = C_n$, the signature of Γ is $(0; an, bn, \underbrace{p, \ldots, p}_{s \text{ times}})$ where a and b are either 1 or p depending upon whether any branch points of π_{C_p} coincide with ramification points of π_K. For such a Γ, the signature of Γ_p is $(0; \underbrace{p, \ldots, p}_{r \text{ times}})$ where*

$$r = sn + \frac{(a-1)|K|}{(p-1)n} + \frac{(b-1)|K|}{(p-1)n}.$$

Proof: In order to determine the signature of Γ, we need to determine the branching data of the map π_Γ. Let R denote the branch points of π_Γ. Since $\pi_\Gamma = \pi_K \circ \pi_{C_p} \circ \pi_\Lambda$, and π_Λ is unramified, any point $\alpha \in R$ has the property that either α is branch point of π_K or is the image of a branch point of π_{C_p} under π_K (or both). If α is branch point of π_K of order m but not the image of a branch point of π_{C_p} under π_K, the branching order of π_Γ at α is m. If α is the image of a branch point of π_{C_p} under π_K but not a branch point of π_K, then the branching order of α is p (since π_{C_p} is either totally ramified of order p or unramified). Finally, if α is both a branch point of π_K of order m and the image of a branch point of π_{C_p}, then it has branching order mp. Hence the possible signatures for Γ are those stated.

To find the signature of Γ_p, we determine the orbits of branch points of π_{C_p} under K. Alternatively, we could apply Proposition 2.2. ∎

5. Previous results and a general form

In this section, we summarize the known results and develop a general form for defining equations for p-gonal surfaces. Let $x\colon X \to \Sigma$ denote the Galois cover of the sphere by the p-gonal group C_p. Let $\mathcal{M}(X)$ denote the function field of X and $\mathbb{C}(x)$ the function field generated by x and the constant functions on X. Classical theory tells us that if $\Phi(x,y)$ is an irreducible polynomial in $\mathbb{C}(x)$ whose splitting field is $\mathcal{M}(X)$, then X is conformally equivalent to the Riemann surface of $\Phi(x,y)$ (see, for example, section 10.9 of [10]). We call $\Phi(x,y)$ a **defining equation** for X. By Kummer theory, there exists $y \in \mathcal{M}(X)$ and $q(x) \in \mathbb{C}(x)$ such that the splitting field of $y^p = q(x)$ is $\mathcal{M}(X)$. In fact, by appropriate choices of x and y, we can guarantee that $q(x)$ is a polynomial, the linear factors each of which have multiplicity bounded between 1 and $p-1$.

Moreover, by construction, the branch points of the map x will be precisely the zeros of the polynomial $q(x)$ (and possibly ∞).

To find the exponents of the linear factors of $q(x)$ we can use the fact that a group of analytic automorphisms of X also acts as a Galois group on the function field of X. The first result we consider gives us a way to relate the exponents of the linear factors of the polynomial $q(x)$ to the quotient map $\varrho_{\Gamma_p}\colon \Gamma_p \to C_p$ (the restriction of $\varrho\colon \Gamma \to G$). Since the problem reduces to a close examination of the action of a generator of C_p on its fixed points, we introduce the following definition.

Definition 5.1: Suppose that a is a branch point of the map π_{C_p}. Then a is the image of the fixed point α of some canonical generator c of Γ_p. We call a the branch point of π_{C_p} corresponding to c.

THEOREM 5.2: *Let $c_1, \dots, c_r$ denote a set of canonical generators of Γ_p. Fix a generator b of C_p such that, as an element of the Galois group $\mathrm{Gal}(\mathbb{C}(x,y)/\mathbb{C}(x))$, $b \cdot y = \zeta y$ where $\zeta = e^{2\pi i/p}$ (we call such an element a* **canonical generator** *for C_p). Let $a_1, \dots, a_r \in \Sigma$ denote the branch points of x corresponding to the canonical generators $c_1, \dots, c_r$ respectively (where a_i could be ∞ for some i). If $\varrho_{\Gamma_p}\colon \Gamma_p \to C_p$ is the quotient map onto the p-gonal group C_p of X and, for $1 \le j \le r$, n_j is the power of b to which c_j is mapped under ϱ_{Γ_p} (so $\varrho_{\Gamma_p}(c_j) = b^{n_j}$), then a defining equation for X is*

$$y^p = \Pi_{j=1}^r (x - a_j)^{n_j}$$

or

$$y^p = \Pi_{j=1}^{i-1} (x - a_j)^{n_j} \Pi_{j=i+1}^{r} (x - a_j)^{n_j}$$

if $a_i = \infty$.

Proof: This was first noted in [6]. For a more detailed exposition, see Section 1 of [4]. ∎

Of course, the construction of these equations depends upon a number of different choices, and these choices will produce different equations. We next present a result which gives complete conditions for determining when two equations constructed using Theorem 5.2 are defining equations for the same p-gonal surface up to conformal equivalence. First we need the following definition.

Definition 5.3: If $q(x)$ is a polynomial, for $M \in \mathrm{PSL}(2,\mathbb{C})$, we define $q_M(x)$ to be the polynomial obtained by applying M to the zeros of $q(x)$.

THEOREM 5.4: *Suppose that $y^p = q(x)$ and $y^p = r(x)$ are defining equations for cyclic p-gonal surfaces X_1 and X_2 constructed using Theorem 5.2. Then X_1 is conformally equivalent to X_2 if and only if $r(x) = (q_M(x))^k$ for some $M \in \mathrm{PSL}(2, \mathbb{C})$ and integer k coprime to p.*

Proof: See [4], Lemma 5.2. ∎

6. Deriving the branch points

If X is a cyclic p-gonal surface, then Theorem 5.2 implies that X has a defining equation of the form $y^p = \Pi_{i=1}^r (x - a_i)^{n_i} = q(x)$ where the a_i are the branch points of the quotient map $\pi_{C_p}\colon X \to X/C_p \cong \Sigma$ for some choice of integers with $1 \leq n_i \leq p - 1$. In general, provided $\Sigma_{i=1}^r n_i \equiv 0 \bmod (p)$, $a_i \neq a_j$ for $i \neq j$, any such choice of a_i's and n_i's will be a defining equation for a cyclic p-gonal surface. However, we are interested in cyclic p-gonal surfaces which admit additional automorphisms as this will allow us to specify the equations further. Therefore, we shall examine the restrictions imposed by the p-gonal normal overgroup on the a_i.

Recall that X/C_p has genus 0 so it can be identified with the Riemann sphere Σ considered as the extended complex plane. Under this identification, the branch points of π_{C_p} will be identified with specific numbers in the extended complex plane and the group K will be identified with a specific finite subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Since the group K acts on the quotient space X/C_p, the branch points of π_{C_p} will fall into K orbits. Applying Theorem 5.2, it follows that the zeros of $q(x)$ will fall into K-orbits.

Henceforth, assume that X/C_p has been identified with Σ via a biholomorphic equivalence and K has consequently been realized as a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ (the implication of such a choice has already been discussed in Theorem 5.4). Let the set $\{a_1, \ldots, a_r\} \subset \Sigma$ be a set of K-orbit representatives of the branch points of π_{C_p}. It is clear that for any a_i, the K-orbit will be the set $\{k(a_i) | k \in S_{a_i})\}$ where S_{a_i} denotes a set of coset representatives for $\mathrm{Stab}_K(a_i)$ in K. The following is an immediate consequence of our observations.

PROPOSITION 6.1: *Suppose X is a cyclic p-gonal surface, C_p a p-gonal group for X and G is the p-gonal normal overgroup for C_p. If $\pi_{C_p}\colon X \to \Sigma$ is the quotient map, $K = G/C_p$ and $\{a_1, \ldots, a_r\}$ are a set of K-orbit representatives for the branch points of π_{C_p}, then there exist integers $n_{i,k}$, $k \in S_{a_i}$ and $1 \leq i \leq r$ such that*

$$y^p = \Pi_{i=1}^r \Pi_{k \in S_{a_i}} (x - k(a_i))^{n_{i,k}}$$

is a defining equation for X.

In order to finish the problem of finding defining equations for a cyclic p-gonal surface X, we need to examine the restrictions imposed by G on the multiplicities of the linear factors of the polynomial $q(x)$. When X is hyperelliptic, the only possible exponent is 1. This means we now have sufficient information to write down defining equations for all hyperelliptic surfaces. Specifically, with the notation we have employed, a defining equation for a hyperelliptic surface X will be

$$y^2 = \Pi_{i=1}^{r}(\Pi_{k\in S_{a_i}}(x - k(\alpha_i))$$

with the obvious minor adjustments if ∞ is a branch point of the quotient map $\pi_{C_2}\colon X \to X/C_2$.

These equations are very general, so to highlight the results we have developed, we shall examine specific examples. In these examples, we assume that the action of K on Σ has been fixed as specified in Appendix A.

Example 6.2: Suppose that Γ is a Fuchsian group with signature $(0; 2, 4, 2g+2)$ for some integer $g \geq 2$. We can define a surface kernel epimorphism $\varrho\colon \Gamma \to V_{2g+2}$ with presentation $\langle x, y | x^4, y^{(2g+2)}, (xy)^2, (x^{-1}y)^2\rangle$ with kernel a surface group for a hyperelliptic surface X of genus g and sphere group D_{2g+2}. Using the results we have developed, a defining equation for such a surface will be $y^2 = \Pi_{i=1}^{2g+2}(x - \zeta^i)$ for some primitive $(2g + 2)$-th root of unity ζ.

Γ	G	K	Ramification	Equation $y^2 = q(x)$
$(0; 2, 2, 2, 2, 2, 2)$	C_2	1	$(0; -)$	$\Pi_{i=1}^{6}(x - \alpha_i)$
$(0; 2, 2, 2, 2, 2)$	V_4	2	$(0; 2, 2, p, p, p)$	$(x^2 - 1)(x^2 - \alpha_1^2)(x^2 - \alpha_2^2)$
$(0; 2, 2, 2, 4)$	D_4	V_4	$(0; 2, 2, 2p, p)$	$x^5 - (\alpha^2 + (\frac{1}{\alpha})^2)x^3 + x$
$(0; 2, 2, 2, 3)$	D_6	S_3	$(0; 2, 2, 3, p)$	$\Pi_{i=1}^{3}(x - \zeta^{2i}\alpha)(x - \frac{\zeta^{2i}}{\alpha})$
$(0; 2, 3, 8)$	$GL(2, 3)$	S_4	$(0; 2, 3, 4p)$	$x^5 - x$
$(0; 2, 5, 10)$	C_{10}	C_5	$(0; 5, 5p, p)$	$x^6 - x$
$(0; 2, 4, 6)$	V_6	D_6	$(0; 2, 2p, 6)$	$x^6 - 1$

Table 2. Equations for genus 2 surfaces.

Example 6.3: In genus 2 every surface is hyperelliptic. Breuer's list for genus 2 (see [2], p. 77) gives us a complete list of signatures for Fuchsian groups containing normal surface groups of orbit genus 2. Applying Theorems 5.1 and 5.2 in [3], we can refine this list so that each entry is a p-gonal normal overgroup for some compact Riemann surface X of genus 2. Using the results we have developed throughout this section, we can write down defining equations for these surfaces. We tabulate them in Table 2. In the equations, the a_i's are pairwise distinct complex numbers. We use i to denote a primitive 4th root of unity and ζ to denote a primitive 6th root of unity.

7. Finding the exponents of the linear factors

To complete the problem of finding defining equations, we need to find the conditions imposed on the multiplicities of the linear factors of $q(x)$ by the action of the p-gonal normal overgroup G of X. For convenience, fix $a_1, \ldots, a_r$, a set of K-orbit representatives for the branch points of π_{C_p}. For a set of canonical generators $c_1, \ldots, c_R$ of Γ_p, let $c_1, \ldots, c_r$ denote a subset with corresponding branch points $a_1, \ldots, a_r$. In order to find the multiplicities for the linear factors of $q(x)$, we shall apply Theorem 5.2 to a special choice of canonical generators for Γ_p related to $c_1, \ldots, c_r$.

Theorem 7.1: *Any set of canonical generators for Γ_p is of the form $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$, $1 \leq i \leq r$, $1 \leq j \leq |K|/|\operatorname{Stab}_K(a_i)|$ where the $\gamma_{i,j}$ have the property that the $\chi(\gamma_{i,j})$ run over the elements in S_{a_i} (a set of coset representatives for $\operatorname{Stab}_K(a_i)$ in K).*

Proof: Consider the first elliptic generator c_1 of Γ_p. By Theorem 2.1, c_1 lies in a unique class $(c^k)^\Gamma$ for some elliptic generator c of Γ. This class splits into $[\Gamma : \Gamma_p \cdot C_\Gamma(c_1)]$ classes in Γ_p and representatives for these classes are $\gamma_i c_1 \gamma_i^{-1}$, where the γ_i run over a set of coset representatives for $\Gamma_p \cdot C_\Gamma(c_1)$ in Γ (for details, see Lemma 3.6 of [2]). Note that if α_1 is the fixed point of c_1 in $\mathbb{H}$, then $\operatorname{Stab}_\Gamma(\alpha_1) = C_\Gamma(c_i)$ and, under χ, the image of $\operatorname{Stab}_\Gamma(\alpha_1)$ is $\operatorname{Stab}_K(a_1)$. It follows that the sets of coset representatives for $\Gamma_p \cdot C_\Gamma(c_1)$ in Γ are precisely the fibers $\chi^{-1}(g)$, where the $g \in K$ run over coset representatives for $\operatorname{Stab}_K(a_i)$ in K. By applying the same argument to the other elliptic generators c_i of Γ_p, we get a set of generators for Γ_p of the form $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$ for $c_1, \ldots, c_r$ as defined above and $\chi(\gamma_{i,j})$, $1 \leq j \leq |K|/|\operatorname{Stab}_K(a_i)|$ running over coset representatives for $\operatorname{Stab}_K(a_i)$ in K. To finish, we need to show that any set of canonical generators for Γ_p is of this form.

For any canonical generator c in a set of canonical generators, since it has finite order, it must be Γ_p-conjugate to a power of one of the generators $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$ defined above. Since $\Gamma_p \lhd \Gamma$, by rewriting in terms of different coset representatives $\gamma_{i,j}$ if necessary, we may assume that $c = \gamma_{i,j} c_i^l \gamma_{i,j}^{-1}$ for some l. By assumption, the local action of c near its fixed point is rotation through $-2\pi/p$. Since every element of $\mathrm{PSL}(2, \mathbb{R})$ is conformal on $\mathbb{H}$, it follows that the action of c near its fixed point will be the same as the action of c_i^l near its fixed point. Again, by assumption, the action of c_i near its fixed point is rotation through $-2\pi/p$ and the action of c_i^l will be rotation through $-2l\pi/p$, so it follows that $l = 1$. ∎

This result motivates the following definition.

Definition 7.2: A subset $c_1, \ldots, c_r$ of canonical generators, with the property that every other element in this set is of the form $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$ as described above, will be called a Γ-set of canonical generators for Γ_p, or a Γ-set for short.

Henceforth, let b denote a canonical generator of C_p and fix $c_1, \ldots, c_r$, a Γ-set corresponding to $a_1, \ldots, a_r$ respectively. Let $n_1, \ldots, n_r$ with $1 \leq n_i \leq p - 1$ be the powers of b to which the generators $c_1, \ldots, c_r$ are mapped under ϱ_Γ respectively. Also, for the Γ-set $c_1, \ldots, c_r$, fix a set of canonical generators $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$ for Γ as given in Theorem 7.1. To apply Theorem 5.2, for each $k \in K$, we need to find the multiplicity of $k(a_i)$ in $q(x)$. This can be done by applying the following easy result.

Lemma 7.3: *For $k \in K$, the canonical generator of Γ_p corresponding to $k(a_i)$ is $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$ where $\chi(\gamma_{i,j}) = k$.*

Proof: We just need to find the image of the fixed point of $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$ in $\mathbb{H}$ under the map π_{Γ_p}. The fixed point of $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$ in $\mathbb{H}$ will be $\gamma_{i,j}(\alpha_i)$, where α_i is the fixed point of c_i in $\mathbb{H}$. Consequently, the branch point corresponding to $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$ in Σ will be $\pi_{\Gamma_p}(\gamma_{i,j}(\alpha_i)) = [\gamma_{i,j}(\alpha_i)]_{\Gamma_p} = \chi(\gamma_{i,j})[\alpha_i]_{\Gamma_p} = \chi(\gamma_{i,j})(a_i) = k(a_i)$. ∎

Recall that $\mathcal{A}$ denotes the action map of K on the p-gonal group C_p. Since $\mathrm{Aut}(C_p)$ is cyclic, there exists $k \in K$ such that $\mathcal{A}(k)$ generates the image of $\mathcal{A}(K)$ in $\mathrm{Aut}(C_p)$. Fix such a k and suppose that N is the integer with $\mathcal{A}(k)(b) = b^N$. Then we get the following nice result.

Lemma 7.4: *Suppose that $\varrho(c_i) = b^{n_i}$ for $1 \leq i \leq r$. Then the image of the elliptic generator $\gamma_{i,j} c_i \gamma_{i,j}^{-1}$ under ϱ_{Γ_p} is $\mathcal{A}(\chi(\gamma_{i,j}))(b^{n_i})$. In particular, for k and N as defined above, if $\chi(\gamma_{i,j}) \in k^T \operatorname{Ker}(\mathcal{A})$ then $\varrho(\gamma_{i,j} c_i \gamma_{i,j}^{-1}) = b^{n_i N^T}$.*

Proof: This is a direct consequence of the results we have developed throughout this section. ∎

We now have all the necessary material to construct equations for cyclic p-gonal surfaces. Note that the statement of Theorem 7.5 holds trivially for $p = 2$, so we do not specify any conditions on p. Before we state the result, we recall the different choices which have been made in order to find $q(x)$ (the implications of these choices have already been discussed in Theorem 5.4).

- X is cyclic p-gonal, C_p is some fixed p-gonal group for X, G is its p-gonal normal overgroup and K is its sphere group.
- Λ is a surface kernel for X and Γ and Γ_p are the Fuchsian groups with $\Gamma/\Lambda = G$ and $\Gamma_p/\Lambda = C_p$.
- $\varrho\colon \Gamma \to G$ and $\chi\colon \Gamma \to K$ are fixed epimorphisms with kernels Λ and Γ_p respectively.
- $\Phi\colon X/C_p \to \Sigma$ is a biholomorphic identification and $a_1, \ldots, a_r$ are a set of K-orbit representatives for the branch points of the map $\Phi \circ \pi_{C_p}$.
- S_{a_i} denotes a set of coset representatives for $\mathrm{Stab}_K(a_i)$ in K.
- $c_1, \ldots, c_r$ are a Γ-set and

$$\{\gamma_{i,j} c_i \gamma_{i,j}^{-1} | 1 \leq i \leq r, 1 \leq j \leq |K|/|Stab_K(a_i)|, \chi(\gamma_{i,j}) \in S(a_i)\}$$

- b is a canonical generator for C_p.
- $k \in K$ is an element whose powers are coset representatives for $\mathrm{Ker}(\mathcal{A})$ in K.
- N is the number with $1 \leq N \leq p-1$ such that $\mathcal{A}(k)(b) = b^N$.

Our results imply the following.

Theorem 7.5: *A defining equation for X is*

$$y^p = \Pi_{i=1}^{r}(\Pi_{j=1}^{n}(\Pi_{g \in k^j \ \mathrm{Ker}(\mathcal{A}) \cap S_{\alpha_i}}(x - g(\alpha_i))^{N^j n_i}).$$

Moreover, if $y^p = q(x)$ is any other defining equation for X satisfying Theorem 5.2, then there exists $M \in \mathrm{PSL}(2, \mathbb{C})$ and $D \in \mathbb{Z}$ such that

$$q(x) = \Pi_{i=1}^{r}(\Pi_{j=1}^{n}(\Pi_{g \in k^j \ \mathrm{Ker}(\mathcal{A}) \cap S_{\alpha_i}}(x - M(g(\alpha_i)))^{DN^j n_i})$$

(where the $DN^j n_i$ are taken modulo p).

8. Examples

To highlight the use of these results and the explicit equations which can be obtained, we finish by looking at a small number examples, deriving classic equations for some very well known surfaces. Unless otherwise stated, we choose an identification of X/C_p with Σ as specified in Appendix A.

Example 8.1: Suppose that Γ has signature $(0; p, p, p)$, and $\varrho\colon \Gamma \to C_p \times C_p$ is a surface kernel epimorphism with kernel Λ of genus $(p-1)(p-2)/2$. Then the surface $X = \mathbb{H}/\Lambda$ is a genus $(p-1)(p-2)/2$ cyclic p-gonal surface. The group Γ_p has signature $(0; \underbrace{p \ldots p}_{p \text{ times}})$, the group $C_p \times C_p$ is a p-gonal normal overgroup and the sphere group for X is $K = C_p$. After identification of X/C_p with Σ, the map π_{C_p} is branched over the p-th roots of unity ζ^i, $1 \leq i \leq p$, where ζ is a primitive pth root of unity. Since the action map of $K = C_p$ on the p-gonal group is trivial, the powers of a canonical generator b of C_p to which a set of canonical generators of Γ_p are mapped will be the same. Therefore, without loss of generality, we assume that $\varrho(c) = b$ for each canonical generator c of Γ_p. Using our results, we get a defining equation for X of the form

$$y^p = \Pi_{i=1}^p (x - \zeta^i) = x^p - 1.$$

If we choose instead the function $z = -x$, this will be branched over $-\zeta^i$ and we get a defining equation for X of the form

$$y^p = \Pi_{i=1}^p (\zeta^i - z) = 1 - z^p$$

or

$$z^p + y^p = 1,$$

which is a defining equation for the pth Fermat curve.

Example 8.2: Suppose that Γ has signature $(0; 4, 4, 5)$ and $\varrho\colon \Gamma \to C_5 \rtimes C_4$ is a surface kernel epimorphism where the action of C_4 on the elements of C_5 is inversion (so the action map has kernel of order 2). If Λ denotes this surface kernel, then $X = \mathbb{H}/\Lambda$ is a genus 4 cyclic 5-gonal surface and $C_5 \rtimes C_4$ is the 5-gonal normal overgroup. The group Γ_5 has signature $(0; 5, 5, 5, 5)$ and if $\{c\} \in \Gamma_5$ is a Γ-set for Γ_5 and $\gamma \in \Gamma$ has order 4, then $\{c, \gamma c \gamma^{-1}, \gamma^2 c \gamma^{-2}, \gamma^3 c \gamma^{-3}\}$ is a set of generators for Γ_5. Note that these are not canonical generators, but are each Γ_p-conjugate to different canonical generators in a set of canonical generators. In particular, the images under ϱ of these elements will be the same as that of a set of canonical generators.

After identification of X/C_p with Σ, the map π_{C_p} is branched over the fourth roots of unity. In fact, we can choose this identification so that the image of γ in the sphere group is the Möbius transformation $M(z) = iz$. Without loss of generality, assume that 1 corresponds to the canonical generator c and that if b is a canonical generator for C_5, then $\mathcal{A}(M)(b) = b^4$ and $\varrho(c) = b$. Since c

corresponds to 1, the exponent of $(x-1)$ in the defining equation for X will be 1. Next, since $M(1) = i$ and $\mathcal{A}(M)(b) = b^4$, it follows that the exponent of $(x-i)$ in the defining equation for X is 4. Likewise, we obtain 1 as the exponent for the term $(x-1)$ and 4 as the exponent of the term $(x+i)$. Therefore, we get a defining equation for X of the form

$$y^5 = (x^2-1)(x^2+1)^4.$$

This surface is actually the Riemann surface of lowest genus on which the group S_5 acts and is called Bring's curve.

Example 8.3: Suppose that Γ has signature $(0;3,3,7)$ and $\varrho\colon \Gamma \to C_7 \rtimes C_3$ is a surface kernel epimorphism with kernel Λ. Then the surface $\mathbb{H}/\Lambda$ is a genus 3 cyclic 7-gonal surface with 7-gonal normal overgroup $C_7 \rtimes C_3$ and sphere group $K = C_3$. The group Γ_7 has signature $(0;7,7,7)$ and if $\{c\} \in \Gamma_7$ is a Γ-set for Γ_7 and $\gamma \in \Gamma$ has order 3, then $\{c, \gamma c\gamma^{-1}, \gamma^2 c\gamma^{-2}\}$ is a set of generators for Γ_7. Instead of identifying X/C_7 with Σ as specified by Appendix A, we choose the identification so that the image of γ in C_3 is the Möbius transformation $M(z) = -1/(z-1)$ and π_{C_p} is branched over 0, 1 and ∞. Without loss of generality, assume that 0 corresponds to the canonical generator c and that if b is a canonical generator for C_7, then $\mathcal{A}(M)(b) = b^2$ and $\varrho(c) = b$. Since c corresponds to 0, the exponent of $(x-0)$ in the defining equation for X will be 1. Next, since $M(0) = 1$ and $\mathcal{A}(M)(b) = b^2$, it follows that the exponent of $(x-1)$ in the defining equation for X is 2. Since the last branch point is ∞, we get

$$y^7 = x(x-1)^2$$

as a defining equation for X. Notice that this is an affine model for Klein's genus 3 surface.

Example 8.4: If we want to find unique defining equations for cyclic prime covers of the Riemann sphere independent of the prime p, we need to classify all surfaces which admit a cyclic p and a cyclic q cover of the Riemann sphere for primes $p \neq q$. This classification, completed in [13], consists of two infinite families and one additional genus 2 surface. Using the results we have developed, we can find different defining equations depending upon the prime used. For example, if X is p-gonal and q-gonal with full automorphism group $C_p \times C_q$ and the normalizer of Λ has signature $(0;pq,p,q)$, our results produce the following two different equations for X:

$$y^p = x^q - 1$$

and

$$y^q = x^p - 1.$$

Appendix A. Actions of K on Σ

Given a cyclic p-gonal surface X, a p-gonal group C_p of X, its normal p-gonal overgroup G, and a surface kernel epimorphism $\varrho\colon \Gamma \to G$ with kernel a surface group for X, the method we have developed to construct defining equations for X produces a $\mathrm{PSL}(2,\mathbb{C})$-class of possible polynomials $q(x)$, each of which is unique up to a power of n coprime to p. The $\mathrm{PSL}(2,\mathbb{C})$-class arises because a defining equation depends upon a choice of identification of X/C_p with Σ. One way to reduce the size of this list is to specify the action of $K = G/C_p$ on Σ and, in particular, specify the generators of K. The following result justifies why we can do this.

Theorem A.1: *Any finite group of conformal automorphisms of the Riemann sphere Σ is isomorphic to C_n, D_n, A_4, S_4 or A_5. Moreover, any two finite groups of automorphisms of Σ of the same isomorphism type are conjugate in the full group of automorphisms of Σ.*

Proof: See [11] Section 1.2. ∎

After identifying X/C_p with Σ via a biholomorphic map Φ, if we compose this map with any Möbius transformation M, this too will yield a biholomorphic equivalence and the group MKM^{-1} as a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ will act on $M(\Sigma)$. By the above theorem, all isomorphism classes are conjugate in $\mathrm{PSL}(2,\mathbb{C})$. In particular, if we can find one representation for a finite group of automorphisms K' of the Riemann sphere in $\mathrm{PSL}(2,\mathbb{C})$, by means of a biholomorphic map, we may identify X/C_p with Σ in such a way that $K = K'$. Specifically, we take $M \in \mathrm{PSL}(2,\mathbb{C})$ where M is the Möbius transformation with $MKM^{-1} = K'$, which exists by the theorem above. The following corollary gives a specific representation of each K in $\mathrm{PSL}(2,\mathbb{C})$. Included is a list of K orbits for points with non-trivial stabilizer for all groups except $K = A_5$. We exclude this case as the orbits are extremely large.

Corollary A.2: *For a cyclic p-gonal surface, we may identify X/C_p with the Riemann sphere (considered as the extended complex plane) in such a way that the group $K = G/C_p$ has generators as tabulated in Table 3. In particular, this is dependent only upon the group K and does not depend upon X. In the table, ζ denotes a primitive n-th root of unity, ω a primitive fifth root of unity, i is* *the usual notation for a primitive fourth root of unity and we have* $1 \leq j \leq n$, $1 \leq l \leq 4$, $1 \leq r, q \leq 5$

Proof: Most of this is a direct consequence of Theorem A.1. For the representations, see Section 1.2 of [11]. The orbits are found by finding the fixed points of each automorphism and their orbits under the action of K. ∎

G	Generators	Orbits of Lengths $< \|K\|$
C_n	$z \to \zeta^j z, 1 \leq i \leq n$	$\{0\}, \{\infty\}$
D_n	$z \to \zeta^i z$ $\zeta^i \frac{1}{z}$	$\{0, \infty\}$, $\{\zeta, \zeta^2, \zeta^3, \ldots, \zeta^n\}$, $\{\zeta^{3/2}, \zeta^{5/2}, \zeta^{7/2}, \ldots, \zeta^{(2n-1)/2}\}$
A_4	$z \to \pm z, \pm\frac{1}{z}, \pm i\frac{z+1}{z-1}$ $\pm i\frac{z-1}{z+1}, \pm\frac{z+i}{z-i}, \pm\frac{z-i}{z+i}$	$\{0, \infty, -1, 1, i, -i\}$ $\{\pm\frac{(i+1)\pm\sqrt{6i}}{2}\}, \{\pm\frac{(1-i)\pm\sqrt{6i}}{2}\}$
S_4	$z \to i^l z, \frac{i^l}{z}, i^l\frac{z+1}{z-1}$ $\pm i^l\frac{z-1}{z+1}, \pm i^l\frac{z+i}{z-i}, \pm i^l\frac{z-i}{z+i}$	$\{0, \infty, -1, 1, i, -i\}, \{\pm\frac{(1\pm i)\pm\sqrt{6i}}{2}\}$ $\{\pm(1 \pm \sqrt{2}), \pm i(1 \pm \sqrt{2}), \pm\frac{\sqrt{2}}{2}(1 \pm i)\}$
A_5	$z \to \omega^r z, -\frac{1}{\omega^r z}$ $\omega^r \frac{-(\omega-\omega^4)\omega^q z+(\omega^2-\omega^3)}{(\omega^2-\omega^3)\omega^q z+(\omega-\omega^4)}$ $\omega^r \frac{-(\omega^2-\omega^3)\omega^q z+(\omega-\omega^4)}{(\omega-\omega^4)\omega^q z-(\omega^2-\omega^3)}$	

Table 3. Standard action for K.

In light of the previous result, by identifying X/C_p with Σ in an appropriate way, we may assume that the group $K = G/C_p$ acts as tabulated in Table 3. We shall call this action the **standard** action of K on Σ. It is natural to ask how many different identifications of X/C_p with Σ there are with the property that K acts standardly. This is important as the number of different identifications will be an upper bound for the number of different polynomials for X our method will produce up to a power coprime to p. The following result which summarizes our discussion specifies precisely the number of different choices we get after fixing this action.

K	$N_{\mathrm{PSL}(2,\mathbb{C})}(K)$	K	$N_{\mathrm{PSL}(2,\mathbb{C})}(K)$	K	$N_{\mathrm{PSL}(2,\mathbb{C})}(K)$
C_n	D_∞	V_4	S_4	$D_n (n \neq 2)$	D_{2n}
A_4	S_4	S_4	S_4	A_5	A_5

Table 4. Normalizers of finite groups of automorphisms of Σ.

COROLLARY A.3: *If X is a cyclic p-gonal surface with normal p-gonal overgroup G and p-gonal group C_p, there exists a biholomorphic map $\Phi: X/C_p \to \Sigma$ such that the group of biholomorphic maps $K = \Phi(G/C_p)\Phi^{-1}$ on Σ acts standardly. This map is unique up to composition with an element in $N_{\mathrm{PSL}(2,\mathbb{C})}(K)$. In particular, if K is specified to act standardly on X, our method of construction will produce up to $|N_{\mathrm{PSL}(2,\mathbb{C})}(K)|$ different defining equations for X where the polynomial $q(x)$ is unique up to a power coprime to p. The groups $N_{\mathrm{PSL}(2,\mathbb{C})}(K)$ are tabulated in Table 4 where $D_\infty = \{z \to \lambda z, z \to 1/z | \lambda \in \mathbb{C}\}$.*

Appendix B. p-gonal normal overgroups

N/C_p	Notation	Presentation	Ker($\mathcal{A}$)
C_n	$C_p \times C_n$	$\langle x, y \| x^p, y^n, [x, y]\rangle$	C_n
C_n	C_{pn}	$\langle x \| x^{pn}\rangle$	C_n
C_n	$C_p \rtimes C_n$	$\langle x, y \| x^p, y^n, x^y = x^a\rangle$	C_k
D_n	$C_p \times D_n$	$\left\langle x, y, z, w \,\middle\|\, x^2, y^2, z^n, xyz, w^p, w^x w^{-1}, w^y w^{-1}, w^z w^{-1} \right\rangle$	D_n
D_n	D_{np}	$\langle x, y, z \| x^2, y^2, z^{np}, xyz\rangle$	C_n
D_n	QD_{np}	$\left\langle x, y, z, w \,\middle\|\, x^2, y^2, z^n, xyz, w^p, w^x w, w^y w, w^z w^{-1} \right\rangle$	C_n
D_n	$C_p \rtimes D_n$	$\left\langle x, y, z, w \,\middle\|\, x^2, y^2, z^n, w^p, xyz, w^x w^{-1}, w^y w, w^z w \right\rangle$	$D_{n/2}$
A_4	$C_p \times A_4$	$\left\langle x, y, z, w \,\middle\|\, x^2, y^3, z^3, w^p, xyz, w^x, w^y, w^z \right\rangle$	A_4
A_4	$C_p \rtimes A_4$	$\left\langle x, y, z, w \,\middle\|\, x^2, y^3, z^3, w^p, xyz, w^x, w^y w^c, w^z w^{-c} \right\rangle$	V_4
A_4	$V_4 \rtimes C_9$	$\left\langle x, y, z \,\middle\|\, x^2, y^2, z^9, x^y x, x^z y, y^z xy \right\rangle$	A_4
S_4	$C_p \times S_4$	$\left\langle x, y, z, w \,\middle\|\, x^2, y^3, z^4, w^p, xyz, w^x w^{-1}, w^y w^{-1}, w^z w^{-1} \right\rangle$	S_4
S_4	$C_p \rtimes S_4$	$\left\langle x, y, z, w \,\middle\|\, x^2, y^3, z^4, w^p, xyz, w^x w, w^y w^{-1}, w^z w \right\rangle$	A_4
S_4	$(V_4 \rtimes C_9) \rtimes C_2$	$\left\langle x, y, z, w \,\middle\|\, x^2, y^2, z^9, w^2, y^x y, x^z y, y^z xy, x^w y, y^w x, z^w z^{-8} \right\rangle$	A_4
A_5	$C_p \times A_5$	$\left\langle x, y, z, w \,\middle\|\, x^2, y^3, z^5, w^p, xyz, w^x w^{-1}, w^y w^{-1}, w^z w^{-1} \right\rangle$	A_5

Table 5. p-gonal normal overgroups.

The following summarizes the notation used in Table 5:

(i) k is the smallest positive integer dividing $(p-1)$ such that $a(k-1) \equiv 0 \bmod (p)$.
(ii) For two group elements g and h, g^h denotes conjugation of g by h.
(iii) V_4 denotes Klein's group of order 4.
(iv) c is an integer with $c^3 \equiv 1 \bmod (p)$ (provided such an integer exists).

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